Mathematical Models of Smart Obstacles

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ABSTRACT

We propose mathematical models to describe the behaviour of smart obstacles. In the context of acoustic scattering a smart obstacle in an obstacle that when hit b y an incoming acoustic wave reacts circulating on its boundary a "pressure current" to pursue a given goal.

A pressure current is a quantity whose physical dimension is pressure divided by time. The goals considered are: 1) to be undetectable, 2) to appear with a shape and an acoustic boundary impedance different from its actual ones, 3) to appear in a location in space different from its actual one eventually with a shape and an acoustic boundary impedance different from its actual ones. He mathematical models proposed for the smart obstacles are optimal control problem for the wave equation. These optimal control problems are studied analytically and solved quantitatively using ad hoc numerical methods.

1. 0 INTRODUCTION

In this paper a mathematical model for an acoustic time dependent scattering problem involving smart obstacles is formulated. Smart obstacles are obstacles that when hit by an incoming acoustic field react in order to pursue an assigned goal. The goal pursued by the smart obstacle considered in this paper is: to appear in a location in space different from its actual location eventually with a shape and boundary impedance different from its actual ones. We call this goal: to appear as a ghost obstacle. The smart obstacle pursues its goal circulating a pressure current (i.e. a quantity whose physical dimension is pressure divided by time) on its boundary. We show that the pressure current necessary to pursue the goal can be determined as the solution of a suitable optimal control problem for the wave equation.

The author and its coworkers have studied similar models for several other classes of smart obstacles in acoustic and electromagnetic scattering (see for example (1)-(6) and the website: http://www.econ.univpm.it/recchioni). The obstacles considered pursue one of the following goals:

1. to be undetectable (i.e.: furtivity problem),
2. to appear with a shape and a boundary impedance different from its actual shape and impedance (i.e.: masking problem),
3. to appear in a location in space different from its actual location eventually with a shape and boundary impedance different from its actual ones (i.e.: ghost obstacle problem).

Zirilli, F. (2006) Mathematical Models of Smart Obstacles. In *Battlefield Acoustic Sensing for ISR Applications* (pp. 20-1 – 20-8). Meeting Proceedings RTO-MP-SET-107, Paper 20. Neuilly-sur-Seine, France: RTO. Available from: http://www.rto.nato.int/abstracts.asp.

Report Documentation Page

Form Approved
OMB No. 0704-0188

1. REPORT DATE **01 OCT 2006**	2. REPORT TYPE **N/A**	3. DATES COVERED **-**

4. TITLE AND SUBTITLE **Mathematical Models of Smart Obstacles**	5a. CONTRACT NUMBER
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER
6. AUTHOR(S)	5d. PROJECT NUMBER
	5e. TASK NUMBER
	5f. WORK UNIT NUMBER
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) **Dipartimento di Matematica G. Castelnuovo Università di Roma La Sapienza, Piazzale Aldo Moro 2 00185 Roma Italy**	8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR’S ACRONYM(S)
	11. SPONSOR/MONITOR’S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release, distribution unlimited

13. SUPPLEMENTARY NOTES
See also ADM202421., The original document contains color images.

14. ABSTRACT

15. SUBJECT TERMS

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT **unclassified**	b. ABSTRACT **unclassified**	c. THIS PAGE **unclassified**	**UU**	**42**	

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std Z39-18

The scattering problems corresponding to 1.-3. have been formulated as optimal control problems for the wave equation (acoustic case) or for the Maxwell equations (electromagnetic case) and the first order optimality conditions for these control problems have been derived applying the Pontryagin maximum principle and solved with appropriate numerical methods on several test problems. The choice of limiting the exposition to the ghost obstacle problem is motivated by the following reasons = necessity to choose a problem to fix the ideas and brevity and the fact that the ghost obstacle problem is relevant in applications and is harder than the furtivity and the masking problems. Several other approaches to study smart obstacles have been considered in the literature, see for example (7)-(10).

2.0 THE GHOST OBSTACLE OPTIMAL CONTROL PROBLEM

Let $\Omega \subset \mathbf{R}^3, \Omega_G \subset \mathbf{R}^3$ be two bounded simply connected open sets with locally Lipschitz boundaries $\partial\Omega, \partial\Omega_G$ and let $\overline{\Omega}$ and $\overline{\Omega}_G$ be their closures respectively. Let us denote with $\underline{n}(\underline{x}) = (n_1(\underline{x}), n_2(\underline{x}), n_3(\underline{x}))^T \in \mathbf{R}^3, \underline{x} \in \partial\Omega$ the outward unit normal vector to $\partial\Omega$ in $\underline{x} \in \partial\Omega$. Since Ω has a locally Lipschitz boundary, $\underline{n}(\underline{x}), \underline{x} \in \partial\Omega$, exists almost everywhere, similar statements hold for the outward unit normal vector to $\partial\Omega_G$. Furthermore let Ω_G be such that $\Omega_G \neq \emptyset$ and $\overline{\Omega} \cap \overline{\Omega}_G = \emptyset$. We assume that Ω and Ω_G are characterized by constant acoustic boundary impedances $\chi \geq 0$ and $\chi_G \geq 0$, respectively. The case $\chi = +\infty$ and/or $\chi_G = +\infty$ (i.e.: the case of acoustically hard obstacles) can be treated with simple modifications of the formulae presented here. We refer to $(\Omega; \chi)$ as the obstacle and to $(\Omega_G; \chi_G)$ as the ghost obstacle. We consider an acoustic incident wave $u^i(\underline{x}, t), (\underline{x}, t) \in \mathbf{R}^3 \times \mathbf{R}$, propagating in a homogeneous isotropic medium in equilibrium at rest with no source terms present that satisfies the wave equation with wave propagation velocity $c > 0$ in $\mathbf{R}^3 \times \mathbf{R}$.

Finally we denote with $u^s(\underline{x}, t), (\underline{x}, t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}$ and with $u^s_G(\underline{x}, t), (\underline{x}, t) \in (\mathbf{R}^3 \setminus \overline{\Omega}_G) \times \mathbf{R}$, the waves scattered respectively by the obstacle $(\Omega; \chi)$ and by the ghost obstacle $(\Omega_G; \chi_G)$ when hit by $u^i(\underline{x}, t), (\underline{x}, t) \in \mathbf{R}^3 \times \mathbf{R}$.

The scattered acoustic field $u^s(\underline{x}, t), (\underline{x}, t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}$ is defined as the solution of the following exterior problem for the wave equation:

$$\Delta u^s(\underline{x}, t) - \frac{1}{c^2} \frac{\partial^2 u^s}{\partial t^2}(\underline{x}, t) = 0, \quad (\underline{x}, t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}, \tag{1}$$

with the boundary condition:

$$-\frac{\partial u^s}{\partial t}(\underline{x}, t) + c\chi \frac{\partial u^s}{\partial \underline{n}(\underline{x})}(\underline{x}, t) = g(\underline{x}, t), \quad (\underline{x}, t) \in \partial\Omega \times \mathbf{R}, \tag{2}$$

where $g(\underline{x}, t)$ is given by:

$$g(\underline{x}, t) = \frac{\partial u^i}{\partial t}(\underline{x}, t) - c\chi \frac{\partial u^i}{\partial \underline{n}(\underline{x})}(\underline{x}, t), \quad (\underline{x}, t) \in \partial\Omega \times \mathbf{R}, \tag{3}$$

the boundary condition at infinity:

$$u^s(\underline{x}, t) = O\left(\frac{1}{r}\right), r \to +\infty, \quad t \in \mathbf{R}, \tag{4}$$

and the radiation condition:

$$\frac{\partial u^s}{\partial r}(\underline{x},t)+\frac{1}{c}\frac{\partial u^s}{\partial t}(\underline{x},t)=o\left(\frac{1}{r}\right),\ r\to+\infty, t\in\mathbf{R}, \tag{5}$$

where $r=\|\underline{x}\|$, $\underline{x}\in R^3, \Delta=\sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2}$ is the Laplace operator, $c>0$ is the wave propagation velocity and $O(\cdot)$ and $o(\cdot)$ are the Landau symbols. We note that $g(\underline{x},t),\ (\underline{x},t)\in\partial\Omega\times\mathbf{R}$ is defined almost everywhere and that the boundary condition (2) can be adapted to deal with the limit case of the acoustically hard obstacles, i.e. $\chi=+\infty$. The obstacle $(\Omega;\chi)$ that scatters the field u^s solution of (1), (2), (3), (4), (5) is called passive obstacle. The field $u_G^s(\underline{x},t),(\underline{x},t)\in\left(\mathbf{R}^3\setminus\overline{\Omega}_G\right)\times\mathbf{R}$ scattered by the (passive) ghost obstacle is defined as the solution of (1), (2), (3), (4), (5) when in the problem defined above we replace Ω with Ω_G and χ with χ_G. Note that we always consider the ghost obstacle as a passive obstacle.
We consider the following problem:
Ghost Obstacle Problem: Given an incoming acoustic field $u^i(\underline{x},t),(\underline{x},t)\in\mathbf{R}^3\times\mathbf{R},$ an obstacle $(\Omega;\chi)$, a ghost obstacle $(\Omega_G;\chi_G)$ choose a pressure current circulating on $\partial\Omega$ for $t\in\mathbf{R}$ in such a way that the wave scattered by $(\Omega;\chi)$ when hit by the incoming acoustic field u^i appears, outside a given set containing Ω and Ω_G, "as similar as possible" to the wave scattered in the same circumstances by the ghost obstacle $(\Omega_G;\chi_G)$.
Remember that a pressure current is a quantity whose physical dimension is: pressure divided by time.
Our goal is to model the ghost obstacle problem as an optimal control problem introducing a control variable $\psi(\underline{x},t),(\underline{x},t)\in\partial\Omega\times\mathbf{R},$ that is a pressure current acting on the boundary of the obstacle. To this aim, we replace the boundary condition (2) with the following boundary condition:

$$-\frac{\partial u^S}{\partial t}(\underline{x},t)+c\chi\frac{\partial u^S}{\partial \underline{n}(\underline{x})}(\underline{x},t)=g(\underline{x},t)+(1+\chi)\psi(\underline{x},t),(\underline{x},t)\in\partial\Omega\times\mathbf{R}. \tag{6}$$

Let Ω_ε be a bounded simply connected open set containing Ω and Ω_G with Lipschitz boundary $\partial\Omega_\varepsilon$ and let $ds_{\Omega_\varepsilon}, ds_{\partial\Omega}$ be the surface measures on $\partial\Omega_\varepsilon$ and $\partial\Omega$ respectively.
We choose the following cost functional:

$$F_{\lambda,\mu,\varepsilon}(\psi)=\int_{\mathbf{R}}dt\left\{\int_{\partial\Omega_\varepsilon}(1+\chi)\lambda\left(u^s(\underline{x},t)-u_G^s(\underline{x},t)\right)^2 ds_{\partial\Omega_\in}+\int_{\partial\Omega}(1+\chi)\mu\varsigma\psi^2(\underline{x},t)ds_{\partial\Omega}\right\}, \tag{7}$$

where $\lambda\geq 0,\mu\geq 0$ are adimensional constants such that $\lambda+\mu=1,$ and ς is a nonzero positive dimensional constant. We model the ghost obstacle problem via the following optimal control problem:

$$\min_{\psi\in C}F_{\lambda,\mu,\varepsilon}(\psi), \tag{8}$$

subject to the constraints (1), (4), (5) and (6).

This is a legitimate mathematical model of the ghost obstacle problem. In fact the minimization $F_{\lambda,\mu,\varepsilon}$ makes small $u^s - u_G^s$ for $(\underline{x},t)\in\partial\Omega_\varepsilon\times\mathbf{R}$, that is makes small $u^s - u_G^s$ for $(\underline{x},t)\in(\mathbf{R}^3\setminus\Omega_\varepsilon)\times\mathbf{R}$ and makes small the "size" of the pressure current used while the constraints (1), (4), (5), (6) guarantee the satisfaction of the dynamic conditions associated to the problem considered.

The set C is the space of the admissible controls that we leave undetermined. The obstacle $(\Omega;\chi)$ that generates the scattered field u^s solution of (8), (1), (4), (5), (6) is called smart or active obstacle.

Note that in (7) the choice $\Omega_G\subset\Omega,\Omega_\varepsilon=\Omega$ gives the masking problem and that the choice $\Omega_G=\phi$, $\Omega_\varepsilon=\Omega$ gives the furtivity problem.

3.0 THE FIRST ORDER OPTIMALITY CONDITIONS

Let us make the following assumptions: let (r,θ,ϕ) be the usual spherical coordinate system in $\mathbf{R}^3$ with pole in the origin, let B be the sphere with center the origin and radius one and let ∂B be its boundary, we assume that:

(a) the boundary of the obstacle Ω is a starlike surface with respect to the origin, that is Ω and ∂B can be represented as follows:

$$\Omega=\left\{\underline{x}=r\underline{\hat{x}}\in\mathbf{R}^3 \middle| 0\le r<\xi(\underline{\hat{x}}),\underline{\hat{x}}\in\partial B\right\}, \tag{9}$$

$$\partial\Omega=\left\{\underline{x}=r\underline{\hat{x}}\in\mathbf{R}^3 \middle| r=\xi(\underline{\hat{x}}),\underline{\hat{x}}\in\partial B\right\}, \tag{10}$$

where $\xi(\underline{\hat{x}})>0,\underline{\hat{x}}\in\partial B$, is a single valued function defined on ∂B that is assumed sufficiently regular for the manipulations that follow;

(b) the sets Ω_ε and $\partial\Omega_\varepsilon$ can be represented as follows:

$$\Omega_\varepsilon=\left\{\underline{x}=r\underline{\hat{x}}\in\mathbf{R}^3 \middle| 0\le r<(\xi(\underline{\hat{x}})+\varepsilon),\underline{\hat{x}}\in\partial B\right\},\varepsilon>0, \tag{11}$$

$$\partial\Omega_\varepsilon=\left\{\underline{x}=r\underline{\hat{x}}\in\mathbf{R}^3 \middle| r=\xi(\underline{\hat{x}})+\varepsilon,\underline{\hat{x}}\in\partial B\right\},\varepsilon>0. \tag{12}$$

for a suitable choice of $\varepsilon > 0$.

Note that the assumptions (*a*) and (*b*) are only one of many other possible choices of assumptions that can be made to guarantee the satisfactory solution of the model (8), (1), (4), (5), (6). This choice is made just to fix the ideas and to keep the exposition simple.

Under the assumptions (*a*) and (*b*), applying the Pontryagin maximum principle the optimal state trajectory $\widetilde{u}^s$ and the corresponding adjoint variable trajectory $\widetilde{\varphi}$ satisfy the necessary first order optimality conditions associated to the optimal control problem (8), (1), (4), (5), (6), that is they are the solution of the following exterior problem for a system of two coupled wave equations:

$$\Delta\widetilde{u}^s(\underline{x},t)-\frac{1}{c^2}\frac{\partial^2\widetilde{u}^s}{\partial t^2}(\underline{x},t)=0,\ (\underline{x},t)\in(\mathbf{R}^3\setminus\overline{\Omega})\times\mathbf{R}, \tag{13}$$

$$\widetilde{u}^s(\underline{x},t)=O\left(\frac{1}{r}\right),\quad r\to+\infty,\ t\in\mathbf{R}, \tag{14}$$

$$\frac{\partial \tilde{u}^s}{\partial r}(\underline{x},t)+\frac{1}{c}\frac{\partial \tilde{u}^s}{\partial t}(\underline{x},t)=o\left(\frac{1}{r}\right),\ r\to+\infty,\ t\in\mathbf{R}, \tag{15}$$

$$\begin{aligned}-\frac{\partial \tilde{u}^s}{\partial r}(\underline{x},t)+c\chi\frac{\partial \tilde{u}^s}{\partial \underline{n}(\underline{x})}(\underline{x},t)=g(\underline{x},t)-\\ -\frac{(1+\chi)}{2\mu\varsigma}\tilde{\varphi}(\underline{x},t),(\underline{x},t)\in\partial\Omega\times\mathbf{R},\end{aligned} \tag{16}$$

$$\Delta\tilde{\varphi}(\underline{x},t)-\frac{1}{c^2}\frac{\partial^2\tilde{\varphi}}{\partial t^2}(\underline{x},t)=0,\ (\underline{x},t)\in\left(\mathbf{R}^3\setminus\overline{\Omega}\right)\times\mathbf{R}, \tag{17}$$

$$\tilde{\varphi}(\underline{x},t)=O\left(\frac{1}{r}\right),\ \ r\to+\infty,\ t\in\mathbf{R}, \tag{18}$$

$$\frac{\partial \tilde{\varphi}}{\partial r}(\underline{x},t)+\frac{1}{c}\frac{\partial \tilde{\varphi}}{\partial t}(\underline{x},t)=o\left(\frac{1}{r}\right),\ r\to+\infty,\ t\in\mathbf{R}, \tag{19}$$

$$-\frac{\partial \tilde{\varphi}}{\partial t}(\underline{x},t)-c\chi\frac{\partial \tilde{\varphi}}{\partial \underline{n}(\underline{x})}(\underline{x})=\ -2\lambda(1+\chi)f_\varepsilon\left(\frac{\underline{x}}{\|\underline{x}\|}\right)\left(\tilde{u}^s\left(x+\varepsilon\frac{\underline{x}}{\|\underline{x}\|},t\right)-u_G^s\left(\underline{x}+\varepsilon\frac{\underline{x}}{\|\underline{x}\|},t\right)\right),(\underline{x},t)\in\partial\Omega\times\mathbf{R}, \tag{20}$$

$$\lim_{t\to-\infty}\tilde{u}^s(\underline{x},t)=0,\underline{x}\in\mathbf{R}^3\setminus\overline{\Omega}, \tag{21}$$

$$\lim_{t\to+\infty}\tilde{\varphi}(\underline{x},t)=0,\underline{x}\in\mathbf{R}^3\setminus\Omega, \tag{22}$$

where $f_\varepsilon\left(\underline{x}/\|\underline{x}\|\right), x\in\partial\Omega$ is the function defined by:

$$\begin{aligned}&f_\varepsilon\left(\frac{\underline{x}}{\|\underline{x}\|}\right)=f_\varepsilon(\hat{\underline{x}}(\theta,\phi))=\frac{\upsilon_\varepsilon(\theta,\phi)}{\upsilon(\theta,\phi)},\underline{x}\in\partial\Omega,\hat{\underline{x}}=\frac{\underline{x}}{\|\underline{x}\|}\in\partial B,\\ &0\le\theta\le\pi,0\le\phi<2\pi,\end{aligned} \tag{23}$$

$$\begin{aligned}&\upsilon(\theta,\phi)=\xi\sqrt{\left(\frac{\partial\xi}{\partial\theta}\right)^2\sin^2\theta+\left(\frac{\partial\xi}{\partial\phi}\right)^2+\xi^2\sin^2\theta},\\ &0\le\theta\le\pi,0\le\phi<2\pi,\end{aligned} \tag{24}$$

$$\begin{aligned}&\upsilon_\varepsilon(\theta,\phi)=(\xi+\varepsilon)\sqrt{\left(\frac{\partial\xi}{\partial\theta}\right)^2\sin^2\theta+\left(\frac{\partial\xi}{\partial\phi}\right)^2+(\xi+\varepsilon)^2\sin^2\theta},\\ &0\le\theta\le\pi,0\le\phi<2\pi.\end{aligned} \tag{25}$$

The relation between $\tilde{\varphi}$ and the optimal control $\tilde{\psi}$ solution of problem (8), (1), (4), (5), (6) is the following one:

$$\tilde{\psi}(\underline{x},t)=-\frac{1}{2\mu\varsigma}\tilde{\varphi}(\underline{x},t),(\underline{x},t)\in\partial\Omega\times\mathbf{R}. \tag{26}$$

Let us point out that we have:

$$ds_{\partial\Omega} = \upsilon(\theta,\phi)d\theta d\phi, 0 \le \theta \le \pi, 0 \le \phi < 2\pi\,, \qquad (27)$$

and

$$ds_{\partial\Omega\varepsilon} = \upsilon_{\varepsilon}(\theta,\phi)d\theta d\phi, 0 \le \theta \le \pi, 0 \le \phi < 2\pi\,, \qquad (28)$$

4.0 NUMERICAL SOLUTION OF THE EXTERIOR PROBLEM (13) – (22)

Numerical methods to solve the exterior problem (13)-(22) have been developed in [5], [6]. These methods belong to the class of the operator expansion methods and are highly parallelizable. Some numerical experiments proving the validity of the control problem proposed as mathematical model of the ghost obstacle problem are shown in the website http://www.econ.univpm.it/recchioni/w11.

5.0 EXTENSION AND CONCLUSIONS

The work presented can be extended to a new class of smart obstacles that pursue the following goal:

4. one of the goals specified in the Introduction restricted to a definite band in the frequency space.

For reasons of brevity we restrict our attention to the definite band ghost obstacle problem in the acoustic case. This problem is formulated as an optimal control problem for the wave equation. The acoustic definite band masking problem and furtivity problem can be treated similarly. We consider the study of these problems as preliminary to the study of the corresponding problems in the electromagnetic case where the wave equation must be replaced with the Maxwell equations.

We note that restricting the goal pursued to a definite band in the frequency space modifies substantially the mathematical formulation of the problems under scrutiny. In fact the optimal control problems used to model problems 1), 2), 3) in particular the cost functionals that must be minimized in order to model appropriately the problem formulated only on the desired band in the frequency space are “nonlocal”. That is the presence of the definite band makes necessary the use of suitable convolutions involving the anti Fourier transform of the characteristic function of the definite frequency band in the definition of the cost functional. Consequently the first order optimality conditions of these new optimal control problems change substantially and cannot be deduced from those derived in [2], [3], [5] and here for the optimal control problems 1), 2), 3). That is the first order optimally conditions are not expressed by two wave equations coupled by local boundary conditions as in [2], [3], [5] and here but the coupling between the two wave equations is given by nonlocal (in time) boundary conditions. As a consequence the way of solving the first order optimality conditions must be changed.

We can conclude that the idea of modelling the smart obstacles using optimal control problems is an interesting idea. Moreover the work developed until now with the model proposed can be profitably extended in several directions such as the study of closed loop controls, finite horizon controls, or the study of inverse problems involving smart obstacles. These are challenging mathematical questions whose solution can be very valuable in practical applications.

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MATHEMATICAL MODELS OF SMART OBSTACLES

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Smart (or active) obstacles are obstacles that when illuminated by an incoming field react actuating a policy in order to pursue an assigned goal. The design of smart obstacles can be improved by the availability of satisfactory mathematical models of them. We propose the use of models based on optimal control problems to describe the behaviour of smart obstacles.

Let us restrict our attention to the context of time dependent and time harmonic acoustic or electromagnetic scattering. In this context the smart obstacle in order to pursue its goal circulates on its boundary a pressure current (acoustic case) or a surface electric current density (electromagnetic case).

The goal pursued by the smart obstacle considered in our work is one of the following:

to be undetectable (furtivity problem),
to appear with a shape and boundary impedance different from its actual ones (masking problem),
to appear in a location in space different from its actual one eventually with a shape and boundary impedance different from its actual ones (ghost obstacle problem),
one of the previous goals restricted to a given band in the frequency space (definite band problems).

These problems are translated in optimal control problems for the wave equation (acoustic case) or for the Maxwell equations (electromagnetic case).

We consider:

1)Direct scattering problems

In this case the optimal control problems associated to the smart obstacles give a way of characterizing and computing the currents needed to pursue the assigned goal as optimal solutions of the mathematical problems considered,

2) Inverse scattering problems

That is starting from some knowledge of "far field data" generated by the smart obstacle in correspondence of several known incoming fields and the knowledge of the "nature" (furtive, masked ghost) of the smart obstacle we want to reconstruct the shape of the obstacle. In this case the optimal control model gives the mathematical framework where to formulate the inverse problem.

The work presented is joint work with:
L. Fatone, F. Mariani, G. Pacelli, M.C. Recchioni.

Direct Scattering Problems

See the papers:

F. Mariani, M.C. Recchioni, F. Zirilli: "The use of the Pontryagin maximum principle in a furtivity problem in time dependent acoustic obstacle scattering", Waves in Random Media 11, (2001), 549-575.

L. Fatone, M.C. Recchioni, F. Zirilli: "A masking problem in time dependent acoustic obstacle scattering", ARLO-Acoustic Research Letters Online 5, Issue 2, (2004), 25-30.

L. Fatone, M.C. Recchioni, F. Zirilli: "Furtivity and masking problems in time dependent electromagnetic obstacle scattering", Journal of Optimization Theory and Applications, 121, (2004), 223-257.

L. Fatone, G. Pacelli, M.C. Recchioni, F. Zirilli: "The use of optimal control methods to study two classes of smart obstacles in time dependent acoustic scattering", to appear in Journal of Engineering Mathematics.

and the websites:

http://www.econ.univpm.it/recchioni/w6

http://www.econ.univpm.it/recchioni/w10

http://www.econ.univpm.it/recchioni/w11

The websites contain animations and virtual reality applications representing data obtained in numerical experiments.

Inverse Scattering Problems

See the papers:

L. Fatone, M.C. Recchioni, A. Scoccia, F. Zirilli: "Direct and inverse scattering problems involving smart obstacles" Journal of Inverse and Ill Posed Problems 13, (2005), 247-257.

L. Fatone, M.C. Recchioni, F. Zirilli: "A method to solve an acoustic inverse scattering problem involving smart obstacles" submitted to Waves in Random and Complex Media.

and the website:
http://www.econ.univpm.it/recchioni/w13

The website contains animations and virtual reality applications representing data obtained in numerical experiments.

For a general reference to our work on scattering see:

http://www.econ.univpm.it/recchioni

We treat furtivity, masking, ghost obstacle and definite band problems in acoustic and electromagnetic obstacle scattering in a similar way.

For simplicity in this talk we begin concentrating on the direct problem in the simplest case: furtivity problems in acoustics.

We consider a furtivity problem in the context of time dependent three dimensional acoustic obstacle scattering.

Let $\Omega \subset \mathbf{R}^3$ be a bounded simply connected open set with locally Lipschitz boundary $\partial\Omega$ and let $\overline{\Omega}$ be its closure. Let $\underline{n}(\underline{x})$ be the outward unit normal vector to $\partial\Omega$ in $\underline{x} \in \partial\Omega$.
We assume that there exists $a > 0$ such that:

$$\overline{B}_a = \left\{ \underline{x} \in \mathbf{R}^3 \middle| \, \|\underline{x}\| \le a \right\} \subset \Omega .$$

We consider acoustic waves propagating in a homogeneous, isotropic medium in equilibrium at rest filling $\mathbf{R}^3 \setminus \Omega$ with no source terms present.

The obstacle Ω is the scatterer that we assume to be characterized by a constant boundary acoustic impedance $\chi > 0$.
The limit cases of acoustically soft obstacles $(\chi = 0)$ and acoustically hard obstacles $(\chi = +\infty)$ can be considered.

Let $u^i(\underline{x},t), (\underline{x},t) \in \mathbf{R}^3 \times \mathbf{R}$ be the incoming acoustic wave, we assume that $u^i(\underline{x},t)$ satisfies:

$$\Delta u^i(\underline{x},t) - \frac{1}{c^2}\frac{\partial^2 u^i}{\partial t^2}(\underline{x},t) = 0, \ (\underline{x},t) \in \mathbf{R}^3 \times \mathbf{R} ,$$

where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ and $c > 0$ is the wave propagation velocity.

When the incoming wave hits the obstacle Ω a scattered wave $u^s(\underline{x},t), (\underline{x},t) \in (\mathbf{R}^3 \setminus \Omega) \times \mathbf{R}$ is generated.

For "passive" obstacles $u^s(\underline{x},t)$ is defined as the solution of the following problem:

(1) $\Delta u^s - \frac{1}{c^2}\frac{\partial^2 u^s}{\partial t^2} = 0, \quad (\underline{x},t) \in (\mathbf{R}^3 \setminus \overline{\Omega}) \times \mathbf{R}$,

(2) $-\frac{\partial u^s}{\partial t} + \mathrm{c}\chi \frac{\partial u^s}{\partial \underline{n}} = g(\underline{x},t), (\underline{x},t) \in \partial\Omega \times \mathbf{R}$,

(3) $u^s(\underline{x},t) = O\left(\frac{1}{r}\right), \quad r = \|\underline{x}\| \to \infty, t \in \mathbf{R}$,

(4) $\frac{\partial u^s}{\partial r} + \frac{1}{c}\frac{\partial u^s}{\partial t} = o\left(\frac{1}{r}\right), r \to \infty, \; t \in \mathbf{R}$,

where

$$g(\underline{x},t)=\frac{\partial u^i}{\partial t}-c\chi\frac{\partial u^i}{\partial \underline{n}},(\underline{x},t)\in\partial\Omega\times\mathbf{R}.$$

The (direct) furtivity problem studied here can be stated as follows: given the incoming wave packet u^i, the obstacle Ω and its acoustic boundary impedance χ choose a control function, in a suitable class of admissible controls, in order to minimize a cost functional that roughly speaking measures the "magnitude" of the scattered wave.
The control function is defined on the boundary of the obstacle $\partial\Omega$ for $t\in\mathbf{R}$.

The presence of this control function changes the nature of the obstacle from being a "passive" obstacle in being an "active" obstacle, that is a "smart" obstacle.

Let C be the space of the admissible controls, we want to solve the following optimal control problem:

(5) $\min_{\psi \in C} F_{\lambda,\mu}(\psi)$,

subject to:

(6) $\Delta u^s - \frac{1}{c^2}\frac{\partial^2 u^s}{\partial t^2} = 0, \quad (\underline{x}, t) \in \left(\mathbf{R}^3 \setminus \overline{\Omega}\right) \times \mathbf{R}$,

(7) $-\frac{\partial u^s}{\partial t} + c\chi \frac{\partial u^s}{\partial \underline{n}} = g(\underline{x}, t) + (1+\chi)\,\psi(\underline{x}, t), (\underline{x}, t) \in \partial\Omega \times \mathbf{R}$,

(8) $u^s(\underline{x}, t) = O\left(\frac{1}{r}\right), \ r \to \infty, t \in \mathbf{R}$,

(9) $\frac{\partial u^s}{\partial r} + \frac{1}{c}\frac{\partial u^s}{\partial t} = o\left(\frac{1}{r}\right), r \to \infty, t \in \mathbf{R}$,

where:

(10) $F_{\lambda,\mu}(\psi) = \int_{\mathbf{R}} dt \int_{\partial\Omega} (1+\chi) \left(\lambda (u^s)^2 + \mu\xi\psi^2 \right) ds_{\partial\Omega}$,

$\lambda \geq 0, \mu \geq 0, \lambda + \mu = 1$, and $\xi > 0$ is a dimensional constant.

Note that λ, μ are adimensional.

Remind that:

$$F_{\lambda,\mu}(\psi)=\int_{\mathbf{R}} dt \int_{\partial\Omega}(1+\chi)\left(\lambda\left(u^s\right)^2+\mu\xi\psi^2\right)ds_{\partial\Omega}.$$

Remarks:

1. u^s depends on ψ via the boundary condition (7),

2. $\lambda=0,\ \mu=1$ corresponds to the "passive" obstacle, in fact in this case $\hat{\psi}\equiv 0$ minimizes $F_{\lambda,\mu}(\psi)$,

3. $\lambda=1,\ \mu=0$ corresponds to the "undetectable" obstacle, in fact in this case $\hat{\psi}=-\frac{1}{1+\chi}g$ gives $u^s=0$ and $F_{\lambda,\mu}(\hat{\psi})=0$. That is the proposed choice of $\hat{\psi}$ minimizes $F_{\lambda,\mu}(\psi)$ and makes the obstacle undetectable (i.e. $u^s=0$).

The functional $F_{\lambda,\mu}(\psi)$ generates interesting problems when $\lambda>0,\ \mu>0, \lambda+\mu=1$.
In this case minimizing (10) we make u^s "small" in $\left(\mathbf{R}^3\setminus\overline{\Omega}\right)\times\mathbf{R}$ (we make the obstacle furtive). The

explicit presence of ψ in (10) (i.e. $\mu > 0$), implies that a cost must be paid to achieve furtivity.

The assumption that the control function ψ is defined on $\partial\Omega \times \mathbf{R}$ is a natural assumption in many applications.

Dimensionally ψ is pressure divided by time so that we call ψ "pressure current".

Let us apply the Pontryagin maximum principle to the previous optimal control problem.
Note that $u^s(\underline{x},t), (\underline{x},t) \in (\mathbf{R}^3 \setminus \Omega) \times \mathbf{R}$ is the "state trajectory" and $\psi(\underline{x},t), (\underline{x},t) \in \partial\Omega \times \mathbf{R}$ is the "control function".
We begin choosing C the set of the admissible controls and the space U that is supposed to contain $u^s|_{\partial\Omega}$.

We define:

$$D_{\mp} = \{F : (\mathbf{R}^3 \setminus \Omega) \times \mathbf{R} \to \mathbf{R} \text{ such that } F|_{\partial\Omega\times\mathbf{R}},$$

$$\frac{\partial F}{\partial \underline{n}}|_{\partial\Omega\times\mathbf{R}}, \frac{\partial F}{\partial t}|_{\partial\Omega\times\mathbf{R}} \in \mathrm{L}^2(\partial\Omega \times \mathbf{R}),$$

F satisfies the wave equation (1), the boundary condition at infinity (3) and

$$\left.\frac{\partial F}{\partial r}+\frac{1}{c}\frac{\partial F}{\partial t}=o\left(\frac{1}{r}\right), r\to\infty,\ t\in\mathbf{R}\right\},$$

$$C=\left\{f\in \mathrm{L}^2(\partial\Omega\times\mathbf{R}), f(\underline{x},t)\in \mathrm{L}^\infty(\partial\Omega), t\in\mathbf{R}\right.$$

such that $\exists F\in D_-$ such that

$$\left. F\big|_{\partial\Omega\times\mathbf{R}}=f,\ \lim_{t\to-\infty} F(\underline{x},t)=0, \underline{x}\in\mathbf{R}^3\setminus\overline{\Omega}\right\},$$

$$U=\left\{f\in \mathrm{L}^2(\partial\Omega\times\mathbf{R}), f(\underline{x},t)\in \mathrm{L}^\infty(\partial\Omega), t\in\mathbf{R}\right.$$

such that $\exists F\in D_+$ such that

$$\left. F\big|_{\partial\Omega\times\mathbf{R}}=f,\ \lim_{t\to-\infty} F(\underline{x},t)=0, \underline{x}\in\mathbf{R}^3\setminus\overline{\Omega}\right\}.$$

Note that C and U are vector spaces.

The Hamiltonian associated to the optimal control problem considered previously is given by:

$$H\left(u^s,\psi,\varphi,\varphi_o,t\right)=\int_{\partial\Omega} ds_{\partial\Omega}(\underline{x})\left\{\ \varphi_o(\underline{x},t)\ (1+\chi)\right.$$

$$\left.\left[\lambda\left(u^s(\underline{x},t)\right)^2+\mu\xi\left(\psi(\underline{x},t)\right)^2\right]+\varphi(\underline{x},t)\ \left[c\chi\frac{\partial u^s}{\partial n}(\underline{x},t)-(1+\chi)\psi(\underline{x},t)-g(\underline{x},t)\right]\right\}.$$

The functions $\varphi(\underline{x},t), \varphi_o(\underline{x},t)$ are called "adjoint variables".

Note that we can choose $\varphi_o \equiv -1$.

Let $u^i(\underline{x},t), (\underline{x},t) \in \mathbf{R}^3 \times \mathbf{R}$ be an incident wave such that $\forall \psi \in C$ the corresponding u^s solution of (6), (7), (8), (9), satisfies the condition:

$$\lim_{t \to -\infty} u^s(\underline{x},t) = 0, \underline{x} \in \mathbf{R}^3 \setminus \overline{\Omega},$$

and $u^s\big|_{\partial\Omega\times\mathbf{R}}, \dfrac{\partial u^s}{\partial t}\Big|_{\partial\Omega\times\mathbf{R}}, \dfrac{\partial u^s}{\partial \underline{n}}\Big|_{\partial\Omega\times\mathbf{R}}, \in L^2(\partial\Omega\times\mathbf{R})$.

Using the Pontryagin maximum principle it can be shown that the optimal solution of the optimal control problem (5), (6), (7), (8), (9) satisfies the following set of equations:

(11) $\Delta u^s - \dfrac{1}{c^2}\dfrac{\partial^2 u^s}{\partial t^2} = 0, \quad (\underline{x},t) \in \left(\mathbf{R}^3 \setminus \overline{\Omega}\right) \times \mathbf{R}$,

(12) $u^s(\underline{x},t) = O\left(\dfrac{1}{r}\right), \; r \to \infty, t \in \mathbf{R}$,

(13) $\dfrac{\partial u^s}{\partial r} + \dfrac{1}{c}\dfrac{\partial u^s}{\partial t} = o\left(\dfrac{1}{r}\right), r \to \infty, t \in \mathbf{R}$,

(14) $-\dfrac{\partial u^s}{\partial t} + c\chi\dfrac{\partial u^s}{\partial n} = g - \dfrac{(1+\chi)}{2\mu\xi}\varphi, (\underline{x},t) \in \partial\Omega \times \mathbf{R}$,

(15) $\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0, \quad (\underline{x},t) \in \left(\mathbf{R}^3 \setminus \overline{\Omega}\right) \times \mathbf{R}$,

(16) $\varphi(\underline{x},t) = O\left(\frac{1}{r}\right), \ r \to \infty, t \in \mathbf{R}$,

(17) $-\frac{\partial \varphi}{\partial r} - \frac{1}{c}\frac{\partial \varphi}{\partial t} = o\left(\frac{1}{r}\right), r \to \infty, t \in \mathbf{R}$,

(18) $-\frac{\partial \varphi}{\partial t} - \mathrm{c}\chi \frac{\partial \varphi}{\partial \underline{n}} = -2\,\lambda\,(1+\chi)\;u^s, (\underline{x},t) \in \partial\Omega \times \mathbf{R}$,

(19) $\lim_{t\to-\infty} u^s = \lim_{t\to+\infty} \varphi = 0, \ \underline{x} \in \mathbf{R}^3 \setminus \overline{\Omega}$.

The relation between φ and the optimal control $\hat{\psi}$ of problem (5), (6), (7), (8), (9) is the following one:

(20) $\hat{\psi}(\underline{x},t) = -\frac{1}{2\mu\xi}\varphi(\underline{x},t), (\underline{x},t) \in \partial\Omega \times \mathbf{R}, \qquad 0 < \mu < 1$.

That is the solution of the optimal control problem (5), (6), (7), (8), (9) can be characterized through (20) via the solution of an exterior problem for two coupled wave equations, that is problem (11), (12), (13), (14), (15), (16), (17), (18), (19).

This fact makes possible to solve the control problem for the smart obstacle approximately at the same computational cost than the cost necessary to solve the scattering problem for the passive obstacle.

Let us consider now masking problems.

Acoustic Masking Problem

Given the incoming acoustic wave packet u^i, the obstacle Ω and its boundary impedance χ, and given an obstacle D such that $\overline{D} \subset \Omega$, with boundary impendance $\chi_a^{'}$, choose a pressure current (i.e. a control function) defined on the boundary of the obstacle $\Omega, \partial\Omega$, for $t \in \mathbf{R}$ in order to minimize a cost functional that measures the "difference" between the wave scattered by Ω and the wave scattered by D when hit by u^i.

Note that the couple $D, \chi_a^{'}$ is called "the mask".

Let u_D^s be the scattered field generated by $D, \chi_a^{'}$ when hit by u^i ($\overline{D} \subset \Omega$).

The masking problem considered can be treated as a furtivity problem with the cost functional given by:

(21) $F_{2\lambda,\mu}(\psi) = \int_{\mathbf{R}} dt \int_{\partial\Omega} (1+\chi) \left[\lambda \left(u^s - u_D^s \right)^2 + \mu \xi \psi^2 \right] ds_{\partial\Omega}$.

Inverse Scattering Problems

We limit our attention to time harmonic inverse scattering problems.

Let $k=\frac{\omega}{c}$ and $u^{s}_{k,\underline{\alpha}}$ be the time harmonic field $u^{s}(\underline{x},t),(\underline{x},t)\in(\mathbf{R}^{3}\setminus\Omega)\times\mathbf{R}$ scattered by the smart obstacle when hit by the time armonic plane wave $u^{i}_{k,\underline{\alpha}}=e^{ik(\underline{x},\underline{\alpha})}$ similarly let $\varphi_{k,\underline{\alpha}}$ be the corresponding time harmonic component of the adjoint variable $\varphi(\underline{x},t),(\underline{x},t)\in(\mathbf{R}^{3}\setminus\Omega)\times\mathbf{R}$ and $u^{s}_{D,k,\underline{\alpha}}$ be the corresponding time harmonic component of the field scattered by the mask $u^{s}_{D}(\underline{x},t),(\underline{x},t)\in(\mathbf{R}^{3}\setminus\Omega)\times\mathbf{R}$.

Arguing as before it can be seen that for $k \in \mathbf{R}, \underline{\alpha} \in \partial B$ the functions $u^{s}_{k,\underline{\alpha}}, \varphi_{k,\underline{\alpha}}$ satisfy the following exterior boundary value problem:

$$\left(\Delta u^{s}_{k,\underline{\alpha}} + k^{2} u^{s}_{k,\underline{\alpha}}\right)(\underline{x}) = 0, \;\; \underline{x} \in \mathbf{R}^{3} \setminus \overline{\Omega}, \tag{22}$$

$$\left(\Delta \varphi_{k,\underline{\alpha}} + k^{2} \varphi_{k,\underline{\alpha}}\right)(\underline{x}) = 0, \;\; \underline{x} \in \mathbf{R}^{3} \setminus \overline{\Omega}, \tag{23}$$

$$\begin{aligned} & iku^{s}_{k,\underline{\alpha}}(\underline{x}) + \chi \frac{\partial u^{s}_{k,\underline{\alpha}}}{\partial \underline{n}(\underline{x})}(\underline{x}) + \frac{1}{\xi}(1+\chi)\varphi_{k,\underline{\alpha}}(\underline{x}) = \\ & = b_{k,\underline{\alpha}}(\underline{x}), \;\; \underline{x} \in \partial\Omega, \end{aligned} \tag{24}$$

$$\begin{aligned} & (1-\lambda)ik\varphi_{k,\underline{\alpha}}(\underline{x}) - (1-\lambda)\chi \frac{\partial \varphi_{k,\underline{\alpha}}}{\partial \underline{n}(\underline{x})}(\underline{x}) + \\ & + \lambda(1+\chi)\left(u^{s}_{k,\underline{\alpha}}(\underline{x}) - u^{s}_{D,k,\underline{\alpha}}(\underline{x})\right) = 0, \;\; \underline{x} \in \partial\Omega, \end{aligned} \tag{25}$$

with the following conditions at infinity:

$$\frac{\partial u^{s}_{k,\underline{\alpha}}(\underline{x})}{\partial r} - iku^{s}_{k,\underline{\alpha}}(\underline{x}) = o\left(\frac{1}{r}\right), \;\; r \to +\infty, \tag{26}$$

$$\frac{\partial \varphi_{k,\underline{\alpha}}(\underline{x})}{\partial r} + ik\varphi_{k,\underline{\alpha}}(\underline{x}) = o\left(\frac{1}{r}\right), \;\; r \to +\infty \tag{27}$$

where

$$b_{k,\underline{\alpha}}(\underline{x}) = -ike^{ik(\underline{x},\underline{\alpha})}(1+\chi(\underline{n}(\underline{x}),\underline{\alpha})), \underline{x} \in \partial\Omega$$

Let $u^s_{k,\underline{\alpha}}(\underline{x})$, $\varphi_{k,\underline{\alpha}}(\underline{x})$, $\underline{x} \in \mathbf{R}^3 \setminus \overline{\Omega}$ be the solution of (22)-(27). The function $\varphi_{k,\underline{\alpha}}(\underline{x})$, $\underline{x} \in \mathbf{R}^3 \setminus \overline{\Omega}$, solution of (22)-(27) is an auxiliary unknown related to the optimal control $\psi(\underline{x}) = \hat{\psi}(\underline{x})$, $\underline{x} \in \partial\Omega$, by the following relation:

$$\hat{\psi}(\underline{x}) = -\frac{1}{\xi}\varphi_{k,\underline{\alpha}}(\underline{x}), \ \underline{x} \in \partial\Omega \tag{28}$$

Due to the first order optimality condition (22)-(27) the solution of the acoustic time harmonic direct masking problem consists in finding the scattered acoustic field $u^s_{k,\underline{\alpha}}$ and the auxiliary function $\varphi_{k,\underline{\alpha}}$ solutions of (22)-(27) and in determining the optimal control function via equation (28).

In particular $u^{s}_{k,\underline{\alpha}}$ is solution of the Helmholtz equation (22) in $\mathbf{R}^3 \setminus \Omega$ and satisfies the Sommerfeld radiation condition (26) at infinity, this implies that:

$$u^{s}_{k,\underline{\alpha}} = \frac{e^{ikr}}{r} F_u\left(\underline{\hat{x}}, k, \underline{\alpha}\right) + O\left(\frac{1}{r^2}\right),\ r \to \infty . \tag{29}$$

The term $F_u\left(\underline{\hat{x}}, k, \underline{\alpha}\right)$ is called "far field" of $u^{s}_{k,\underline{\alpha}}$.

Similarly $\overline{\varphi}_{k,\underline{\alpha}}$ ($\overline{\cdot}$ = complex conjugate) is solution of the Helmholtz equation (23) in $\mathbf{R}^3 \setminus \Omega$ and satisfies the Sommerfeld radiation condition ($\overline{26}$) ((26) complex conjugate), this implies that:

$$\overline{\varphi}_{k,\underline{\alpha}} = \frac{e^{ikr}}{r} F_{\overline{\varphi}}\left(\underline{\hat{x}}, k, \underline{\alpha}\right) + O\left(\frac{1}{r^2}\right), r \to \infty \tag{30}$$

The term $F_{\overline{\varphi}}\left(\underline{\hat{x}}, k, \underline{\alpha}\right)$ is called "far field" of $\overline{\varphi}_{k,\underline{\alpha}}$.

Acoustic Time Harmonic Masking Inverse Scattering Problem

From the knowledge of several far fields generated by the masked obstacle when hit by known incident acoustic waves when the optimal pressure current is active and of the acoustic boundary impedances of the obstacle and of the mask find the shape of the obstacle.

We generalize to the case of smart (masked) obstacles the Herglotz function method used to reconstruct the shape of an obstacle from far field data in the case of passive obstacles.
This method has been introduced in:

D. Colton and P. Monk, “The numerical solution of the three dimensional inverse scattering problem for time harmonic acoustic waves”, SIAM Journal on Scientific and Statistical Computing, 8, (1987), 193-200, and developed by several authors including:

L. Misici, F. Zirilli, “Three-dimensional inverse obstacle scattering for time harmonic waves: a numerical metnod”, SIAM Journal on Scientific Computing, 15, (1994), 1174-1189.

Let $\boldsymbol{E}$ be the set of the eigenvalues of the matrix Laplace operator $\boldsymbol{\Delta} = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}$ acting on two dimensional vector functions defined in the domain Ω with the homogeneous boundary condition corresponding the to boundary condition (32) on $\partial\Omega$.

Definition 1 Given - $k^2 \notin \mathbf{E}$ let $\underline{W}_k$ be the unique solution of the equation:

$$\left(\boldsymbol{\Delta}\underline{W}_k\right)\left(\underline{y}\right) + k^2 \underline{W}_k\left(\underline{y}\right) = \underline{0}, \quad \underline{y} \in \Omega \tag{31}$$

with the boundary condition:

$$\underline{W}_k\left(\underline{y}\right) - \frac{\chi}{ik}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\frac{\partial \underline{W}_k}{\partial \underline{n}\left(\underline{y}\right)}\left(\underline{y}\right) = \frac{4\pi}{k}\begin{pmatrix} \overline{\Phi}_k\left(\underline{0}, \underline{y}\right) \\ \overline{\Phi}_k\left(\underline{0}, \underline{y}\right) \end{pmatrix} +$$

$$+ \frac{4\pi\chi}{ik^2}\begin{pmatrix} \dfrac{\partial \overline{\Phi}_k\left(\underline{y}\right)}{\partial \underline{n}\left(\underline{y}\right)} \\ 0 \end{pmatrix}, \quad \underline{y} \in \partial\Omega, \tag{32}$$

where $\Phi_k\left(\underline{x}, \underline{y}\right) = \dfrac{e^{ik\left\|\underline{x}-\underline{y}\right\|}}{4\pi\left\|\underline{x}-\underline{y}\right\|}$.

We say that Ω is a smart Herglotz domain at wave number k with respect to the boundary condition (32) if there exists $\underline{\tilde{g}}_k(\hat{\underline{x}}) = \underline{g}_k(\hat{\underline{x}}), \hat{\underline{x}} \in \partial B$ such that:

$$\underline{g}_k(\hat{\underline{x}}) = \begin{pmatrix} g_{k,u}(\hat{\underline{x}}) \\ g_{k,\varphi}(\hat{\underline{x}}) \end{pmatrix}, \ \hat{\underline{x}} \in \partial B, \tag{33}$$

and

$$\underline{W}_k(\underline{y}) = \int_{\partial B} ds_{\partial B}(\hat{\underline{x}}) e^{ik(\hat{\underline{x}}, \underline{y})} \underline{g}_k(\hat{\underline{x}}), \ \underline{y} \in \overline{\Omega}.$$

It is easy to see that the class of the smart Herglotz domains at wave number k is not empty in fact for example, when $-k^2 \notin \mathbf{E}$ the passive sphere is a smart Herglotz domain.

Let us define:

$$\underline{F}\left(\underline{\hat{x}}, k, \underline{\alpha}\right) = \begin{pmatrix} F_{0,u^s}\left(\underline{\hat{x}}, k, \underline{\alpha}\right) \\ \overline{F}_{0,\overline{\varphi}}\left(\underline{\hat{x}}, k, \underline{\alpha}\right) \end{pmatrix}, \ \underline{\hat{x}} \in \partial B, \ \underline{\alpha} \in \partial B, \qquad (35)$$

and for $a, b \in \mathbf{R}$ let

$$\underline{r}_{k,a,b}\left(\underline{\hat{x}}\right) = \begin{pmatrix} a g_{k,u}\left(\underline{\hat{x}}\right) \\ b g_{k,\varphi}\left(\underline{\hat{x}}\right) \end{pmatrix}, \ \underline{\hat{x}} \in \partial B, \qquad (36)$$

Now we can describe the numerical method that we propose to solve the inverse scattering problem involving smart obstacles considered here.

Let $N_1, N_2, N_3, N_4,$ be four positive integers, let $\sum_1 = \left\{ k_i > 0 \,|\, -k_i^2 \notin \mathbf{E}\ , i = 1,2,\ldots, N_1 \right\}$ be the set of the wave numbers considered, $\sum_2 = \left\{ \underline{\alpha}_i \in \partial B,\ i = 1,2,\ldots, N_2 \right\}$ be the set of propagation directions of the incoming waves considered, $\sum_3 = \left\{ \underline{\hat{x}}_i \in \partial B,\ i = 1,2,\ldots, N_3 \right\}$ be the set of directions where the far fields generated by the incoming waves are measured and $\sum_4 = \left\{ \underline{\hat{x}}_i \in \partial B,\ i = 1,2,\ldots, N_4 \right\}$ be a second set of directions where the far fields generated by the incoming waves are measured. Moreover let $\sum_3$ and $\sum_4$ be such that $\sum_3 \cap \sum_4 = \emptyset$.

Let N_λ be a positive integer and let us choose $\lambda_i \in (0,1)$, $i = 1,2\ldots,N_\lambda$ such that $\lambda_i < \lambda_{i+1}$, $i = 1,2,\ldots,N_\lambda - 1$, the numerical algorithm that we propose to reconstruct $\partial\Omega$ is based on the following iterative procedure: let $i = 1$

i_0) set $\lambda = \lambda_i$;

i_1) choose an initial approximation $\partial\Omega^*$ of the boundary $\partial\Omega$;

i_2) using the current approximation $\partial\Omega^*$ of $\partial\Omega$ from the knowledge of the value given to the parameter λ appearing in the first order optimally conditions (22)-(27) and the knowledge of the measures of the far fields $F_{0,u^s}(\underline{\hat{x}},k,\underline{\alpha}), k \in \Sigma_1, \underline{\alpha} \in \Sigma_2, \underline{\hat{x}} \in \Sigma_3 \cup \Sigma_4$ approximate the far field $F_{0,u^s}(\underline{\hat{x}},k,\underline{\alpha}), \overline{F}_{0,\overline{\varphi}}(\underline{\hat{x}},k,\underline{\alpha}), k \in \Sigma_1, \underline{\alpha} \in \Sigma_2, \underline{\hat{x}} \in \partial B$;

i_3) from the relation:

$$a\frac{1}{k_i} = \int_{\partial B} ds_{\partial B}(\underline{\hat{x}})\left(\underline{F}\left(\underline{\hat{x}},k_i,\underline{\alpha}_j\right), \underline{r}_{k_i,a,b}(\underline{\hat{x}})\right),\ i = 1,2,\ldots,N_1, \tag{36}$$
$$j = 1,2,\ldots,N_2, a,b \in \mathbf{R},$$

and from the knowledge of the far fields associated to $u^s_{k,\underline{\alpha}}$ and $\varphi_{k,\underline{\alpha}}$ reconstruct an approximation of the vector Herglotz kernels $\underline{g}_{k_i}$, $i = 1,2,\ldots,N_1$ associated to the smart Herglotz domain Ω;

i_4) from equation (34) and the knowledge of the approximate Herglotz kernels reconstruct the associated vector Herglotz wave functions $\underline{W}_{k_i}$, $i = 1,2,\ldots, N_1$;

i_5) from equation (32) and from the knowledge of the vector Herglotz wave functions $\underline{W}_{k_i}$, $i = 1,2,\ldots, N_1$ reconstruct a new approximation $\partial\widetilde{\Omega}$ of the boundary $\partial\Omega$ of the smart obstacle;

i_6) if the "distance" between the old and the new approximation of $\partial\Omega$, that is the "distance" between $\partial\Omega^*$ and $\partial\widetilde{\Omega}$ is "small" set $\partial\Omega_i = \partial\widetilde{\Omega}$ and go to Step i_7) otherwise set $\partial\Omega^* = \partial\widetilde{\Omega}$ and go to Step i_2);

i_7) if $i = N_\lambda$ stop otherwise set $i = i+1$ and go to Step i_0).

The reconstruction of the couple (λ,Ω) chosen among the solutions (λ_i,Ω_i), $i = 1,2,\ldots, N_\lambda$ obtained with the previous procedure is the one where the boundary condition (32) is satisfied more accurately. That is if the boundary condition (32) is imposed in the least squares sense the one that minimizes the remainder at the end of the least squares procedure.

The first experiment shown, involves a furtive acoustically soft (i.e. $\chi = 0$) double cone (see Fig. 1). The double cone is obtained placing two cones of the same height and base one upon the other with their bases in common and its boundary is defined by:

$$r = f_1(\theta) = \begin{cases} \sin(\pi/4)/\sin(3\pi/4-\theta), & 0 \le \theta \le \pi/2, 0 \le \phi < 2\pi, \\ \sin(\pi/4)/\sin(5\pi/4-\theta), & \pi/2 \le \theta \le \pi, 0 \le \phi < 2\pi, \end{cases} \quad (37)$$

It is easy to see that $f_1(\theta)$ is the sum of an infinite Fourier series. Figure 1 shows the reconstructions obtained applying the procedure described in Step i_0)-i_7) when we choose $\Omega^* = B,\ c = 1,\ \lambda = 0.1$ or $\lambda = 0.9,\ \mu = 1-\lambda,\ \xi = 1,\ L_{\max} = 8,\ \in = 5\cdot 10^{-3},\ L_S = 8$ or $L_S = 20$, $\Sigma_1 = \{4\}, \Sigma_2 = \{\underline{\alpha}_i = (\sin(i\pi/10), 0, \cos(i\pi/10))^T \in \partial B,\ i = 1,2,...,10\}$, $\Sigma_3 = \{\underline{\hat{x}} = (\sin(i\pi/10)\cos(j\pi/5), \sin(i\pi/10)\sin(j\pi/5), \cos(i\pi/10))^T$ and $i = 1,2,...,9,\ j = 1,2,...,10\} \cup \{(0,0,1)^T, (0,0,-1)^T\}$ $\Sigma_4 = \{\underline{\hat{x}} = (\sin((2i-1)\pi/20)\cos((2j-1)\pi/10), \sin((2i-1)\pi/20)$ $\sin((2j-1)\pi/10), \cos((2i-1)\pi/20)^T,\ i = 1,2,...,10,\ j = 1,2,...,10\}$.

We note that the reconstructions shown in Figure 1 are very similar and they depend essentially only on the value of L_S used as shown by the reconstructions obtained when $\lambda = 0.9$ and $L_S = 8$ or $L_S = 20$.

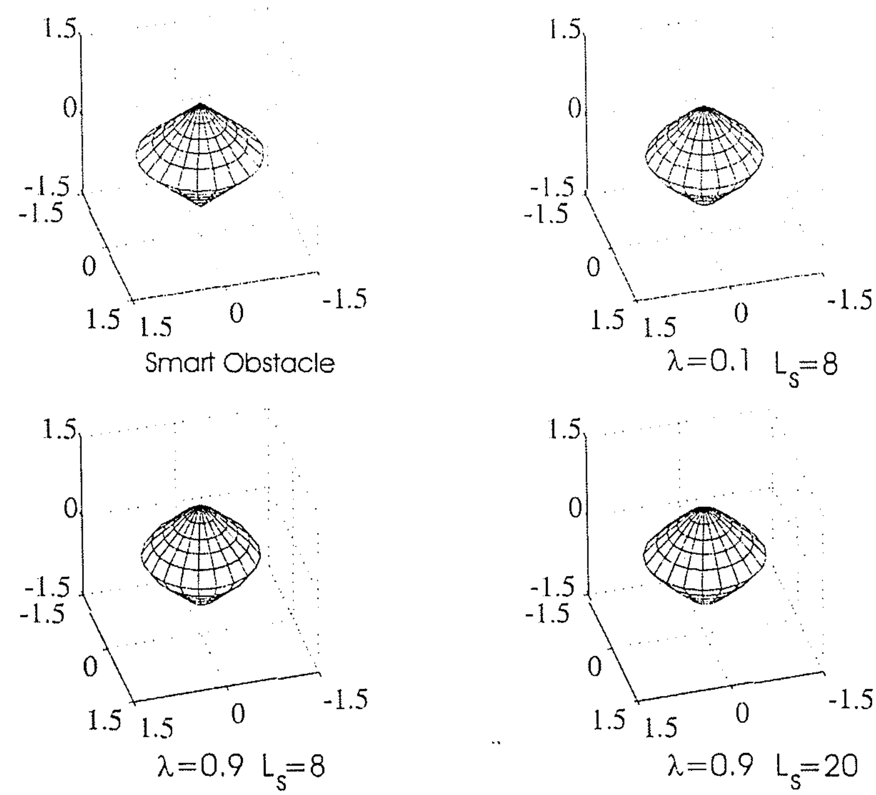

Figure 1: Furtive smart double cone and three reconstructions versus λ and L_s

The second experiment involves a smart acoustically soft (i.e. $\chi = 0$) corrugated sphere whose boundary is given by $r = f(\theta) = 1 - 0.15\sin^2(4\theta), 0 \le \theta \le \pi, 0 \le \phi < 2\pi$ (see Fig. 2)) that pursues the goal of appearing as an acoustically soft (i.e $\chi'_a = 0$) double cone (see Fig. 1 up left corner)). In this experiment the far field data used for the reconstruction are synthetic data obtained solving numerically the corresponding direct scattering problems and have 3-4 significant digits exact. We have chosen the initial guess of $\partial\Omega^*$ assigned in *Step* i_1) to be the surface of the sphere with center the origin and radius 1.2, moreover we have chosen $c = 1, \lambda = 0.1$ (Figure 2), $\lambda = 0.9$ (Figure 3), $\mu = 1 - \lambda, \xi = 1, L_{\max} = L_g = 8, \in = 5 \cdot 10^{-3}$, $L_s = 8$, $\Sigma_1 = \{3\}$, $\Sigma_2 = \{\underline{\alpha}_i = (\sin(i\pi/10), 0, \cos(i\pi/10))^T \in \partial B,\ i = 1,2,\ldots,10\}$, $\Sigma_3 = \{\underline{\hat{x}} = (\sin(i\pi/10)\cos(j\pi/5), \sin(i\pi/10)\sin(j\pi/5), \cos(i\pi/10))^T$ and $i = 1,2,\ldots,9,\ j = 1,2,\ldots,10\} \cup \{(0,0,1)^T, (0,0,-1)^T\}$ and $\Sigma_4 = \{\underline{\hat{x}} = (\sin((2i-1)\pi/20)\cos((2j-1)\pi/10), \sin((2i-1)\pi/20)$. $\sin((2j-1)\pi/10), \cos((2i-1)\pi/20)^T,\ i = 1,2,,\ldots,10,\ j = 1,2,\ldots,10\}$. Figures 2 and 3 show the results obtained applying the procedure to solve the inverse problem i_0)-i_7) when $\lambda = 0.1$ (Figure 2) and $\lambda = 0.9$ (Figure 3). As shown in the figures we choose different values of the parameters *a* and *b* of the initial guess $\partial\Omega^*$.

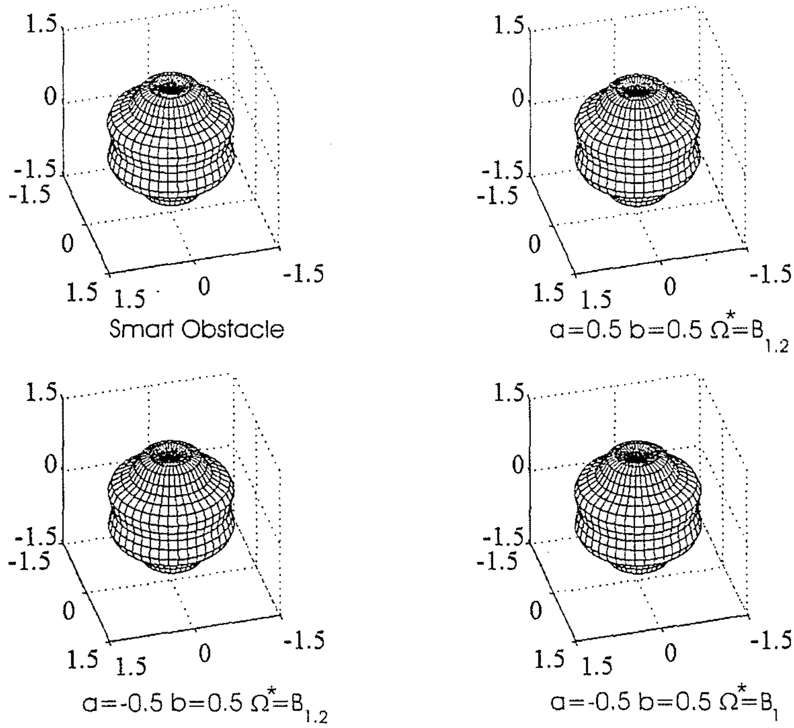

Figure 2: Masked smart obstacle and three reconstructions when $\lambda = 0.1$

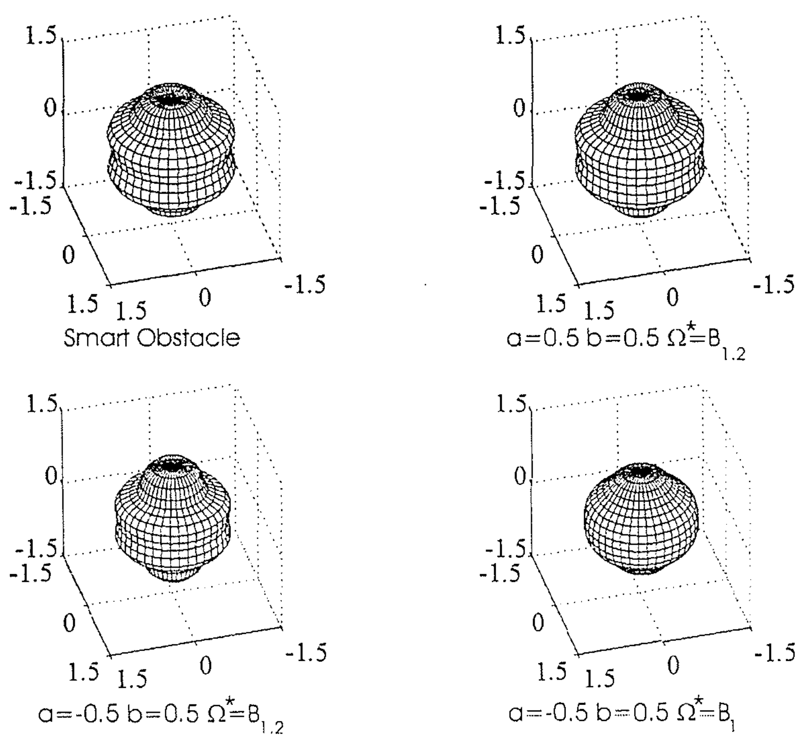

Figure 3: Masked smart obstacle and three reconstructions when $\lambda = 0.9$